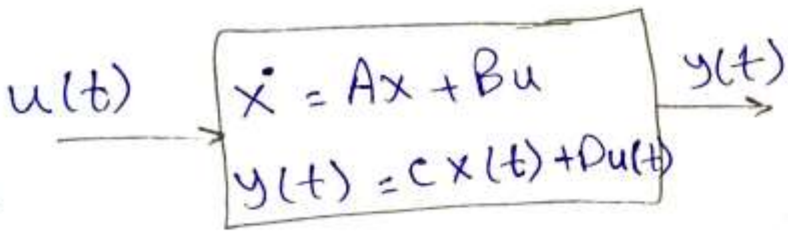
Digital Control

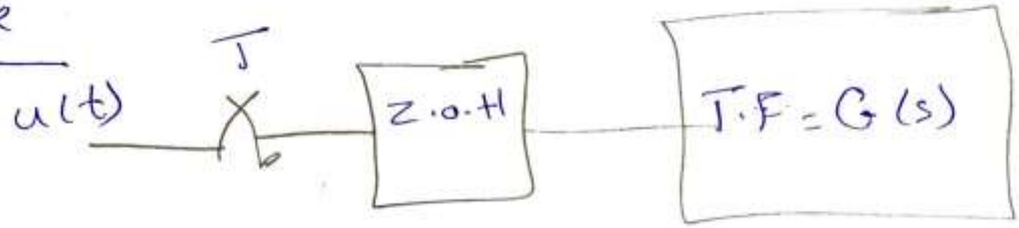
* The Corresponding state-space representation of a Continous system in the discrete time domain:-

In Contn. time system



$$T.F = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

In discrete



$$x(K+1) = A_d\, x(KT) + B_d\, u(t)$$

$$y(KT) = C_d\, X(KT) + D_d\, u(t)$$

$$\text{Pulse } T.F = \mathcal{Z}\left[\overset{1-z^{-1}}{\frac{1-e^{-Ts}}{s}} \cdot \overset{T.F}{G(s)}\right]$$

* Prove the following

$$A_d = \phi(T)$$

$$B_d = \int_0^T \phi(q)\, B\, d_q$$

$$C_d = C \qquad D_d = D$$

given

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}}$$

sampler + Z.O.H

$\hookrightarrow$ discretization

$$\boxed{\begin{aligned} x(K+1)T &= A_d\, x(KT) + B_d\, u(KT) \\ y(KT) &= C_d\, x(KT) + D_d\, u(KT) \end{aligned}}$$

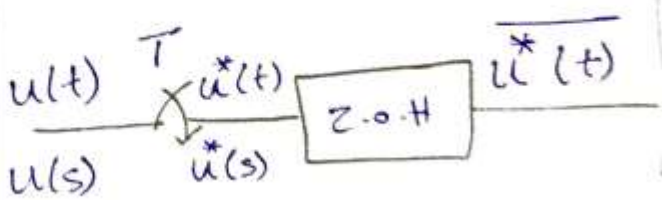
$$y(t) = C\,x(t) + D\,u(t) \xrightarrow{t=KT} y(KT) = C_d\, x(KT) + D_d\, u(KT)$$

$$\boxed{C = C_d \quad , \quad D = D_d}$$

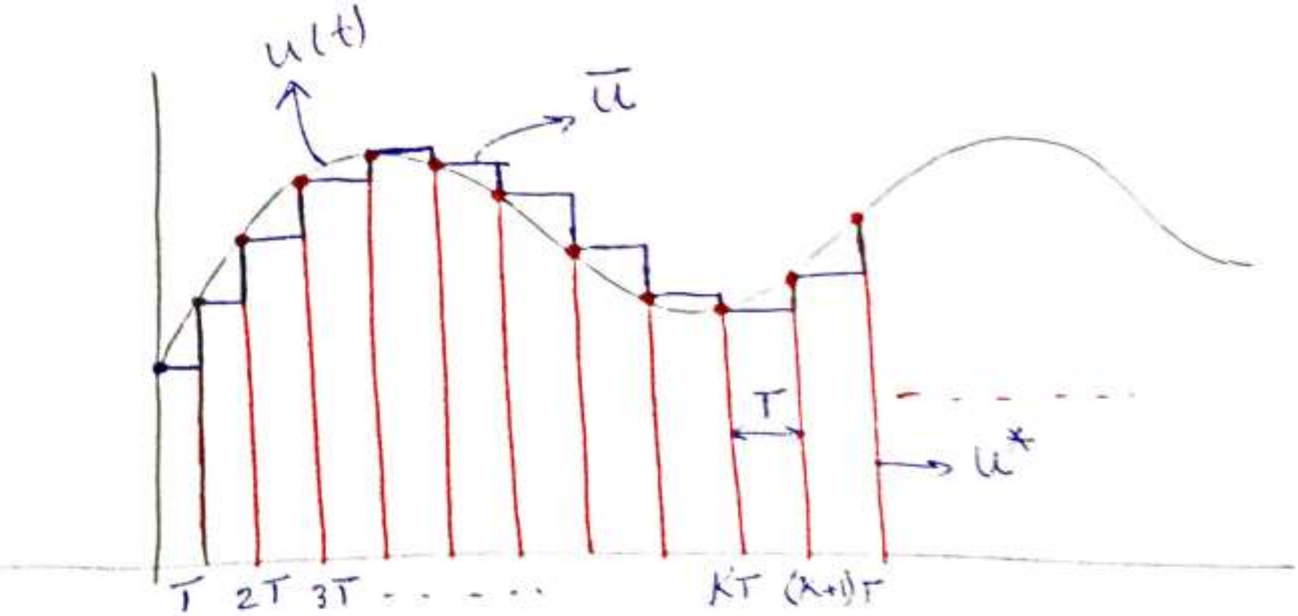
$$\boxed{x(t) = \phi(t)\, \underbrace{x(0)}_{\text{initial time}} + \int_0^t \phi(t-\tau) B u(\tau)\, d\tau}$$

initial time t_0

$$x(t) = \overset{\phi(t-t_0)}{\cancel{\phi(t)}}\, x(t_0) + \int_{t_0}^{t} \phi(t-\tau) B u(\tau)\, d\tau$$



$$\dot{x} = A x(t) + B u(t)$$
$$y = C x(t) + D u(t)$$



$t_0 \longrightarrow$ initial value KT

$t \longrightarrow (K+1)T$

$$\phi(t - t_0) = KT + \overset{T}{\cancel{K}} - KT = T$$

$$x(K+1)T = \phi(T)\,x(KT) + \int_{\tau=KT}^{(K+1)T} \phi\big((K+1)T-\tau\big)\,B\,\underbrace{u(\tau)}_{\text{Const.}}\,d\tau$$

$$x(K+1)T = \phi(T)\,x(KT) + u(KT)\int_{\tau=KT}^{(K+1)T} \phi\big(\underbrace{(K+1)T-\tau}_{q}\big)\,B\,d\tau$$

assume $\boxed{(K+1)T-\tau = q} \rightarrow (1)$

$$\frac{dq}{d\tau} = -1 \rightarrow \boxed{d\tau = -dq}$$

in (1) $\tau = KT \Rightarrow q = T$

$\tau = (K+1)T \Rightarrow q = 0$

$$x(K+1)T = \phi(T)\,X(KT) + U(KT)\int_{T}^{0} \phi(q)\,B * -dq$$

$$x(K+1)T = \phi(T)\,x(KT) + u(KT)\int_0^T \phi(q)\cdot B\cdot dq$$

Compare with

$$x(K+1)T = A_d\,x(KT) + B_d\,u(KT)$$

$$\boxed{\begin{aligned} A_d &= \phi(T) \\ B_d &= \int_0^T \phi(q)\cdot B\cdot dq \end{aligned}}$$

$\rightarrow$ Summary

$$\boxed{\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}}$$

$\Big\downarrow$ Discretization (sampler + Z.O.H)

$$\boxed{\begin{aligned} x(K+1)T &= A_d\,x(KT) + B_d\,u(KT) \\ y(KT) &= C_d\,x(KT) + D_d\,u(KT) \end{aligned}}$$

$$A_d = \phi(T)$$

$$C_d = C \qquad D_d = D$$

$$B_d = \int_0^T \phi(q) \cdot B \cdot dq$$

<u>where:</u>

$$\phi(T) = \phi(t)\Big|_{t=T}$$

$$\phi(t) = \mathcal{L}^{-1}(sI - A)^{-1}$$

$$\phi(q) = \phi(t)\Big|_{t=q}$$

$\boxed{EX}$ find the discrete state-space for:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

sampler and Z.O.H are used with $T = 0.1$ sec.

$$(sI - A) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 0 & s+1 \end{pmatrix}$$

$$|sI - A|^{-1} = \frac{1}{s(s+1)} \begin{pmatrix} s+1 & 1 \\ 0 & s \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{pmatrix}$$

$$(sI - A)^{-1} = \begin{pmatrix} \frac{1}{s} & \frac{1}{s} - \frac{1}{s+1} \\ 0 & \frac{1}{s+1} \end{pmatrix}$$

$$\phi(t) = \mathcal{L}^{-1} (sI - A)^{-1} = \begin{pmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{pmatrix}$$

↳ state-transition matrix

$$A_d = \phi(T) = \phi(t) \Big|_{t=T}$$

$$= \begin{pmatrix} 1 & 1 - e^{-T} \\ 0 & e^{-T} \end{pmatrix}$$

$$A_d = \begin{pmatrix} 1 & 1-e^{-0.1} \\ 0 & e^{-0.1} \end{pmatrix} = \begin{pmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{pmatrix}$$

$$B_d = \int_0^T \phi(q)\, B\, dq$$

$$= \int_0^{0.1} \begin{pmatrix} 1 & 1-e^{-q} \\ 0 & e^{-q} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} dq$$

$$= \int_0^{0.1} \begin{pmatrix} 1-e^{-q} \\ e^{-q} \end{pmatrix} dq = \begin{pmatrix} q+e^{-q} \\ -e^{-q} \end{pmatrix} \Bigg|_0^{0.1}$$

$$B_d = \begin{pmatrix} 0.1+e^{-0.1} \\ -e^{-0.1} \end{pmatrix} - \begin{pmatrix} 0+1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.005 \\ 0.095 \end{pmatrix}$$

$$C_d = C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

$$x(K+1)T = \begin{pmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{pmatrix} x(KT) + \begin{pmatrix} 0.005 \\ 0.095 \end{pmatrix} u(KT)$$

$$y(KT) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(KT)$$

T · F

<u>T · F</u>

$$\text{Pulse T.F} = C_d (zI - A_d)^{-1} B_d + D_d$$

<u>or</u>

$$T.F = C(sI - A)^{-1} B + D = F(s)$$

$$\text{Pulse T.F} = \mathcal{Z}\left[\frac{\overset{(1-z^{-1})}{1 - e^{-Ts}}}{s} \cdot F(s)\right]$$

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* state-Feedback Control
  ↳ (Pole Placement design)

Control design

classical Control Design

- PI
- PD
- PID
- Phase lead, Phase lag ...

modern Control design

⇝ state Feedback control

observer design

Control Problem

regulation control Problem

$(r=0)$

→ we Concern with the disturbance or noise reJection.

→ there is no inPut.

ex state Feedback Control.

Servo Problem

$(r \neq 0)$

→ We Concern with enhancement of the system Performance in the Presence of a desired refrence input

→ inPut is eXisted.

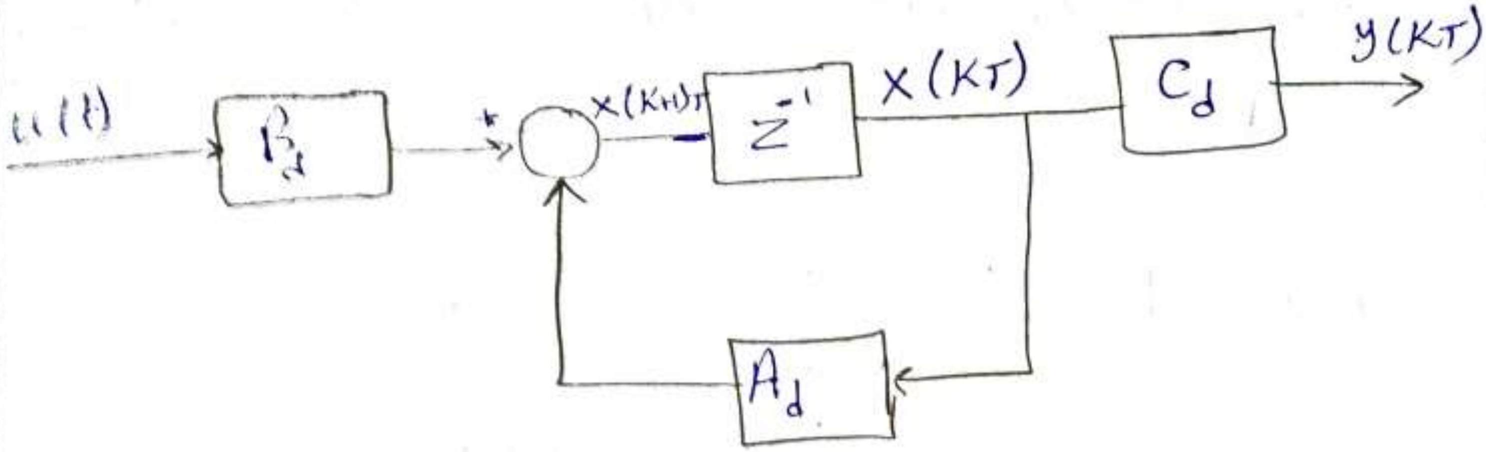
→ For a discrete time System

~~x(KT)~~

$$x(K+1)T = A \times x(KT) + B\,u(t)$$

$$y(KT) = C\,x(KT)$$

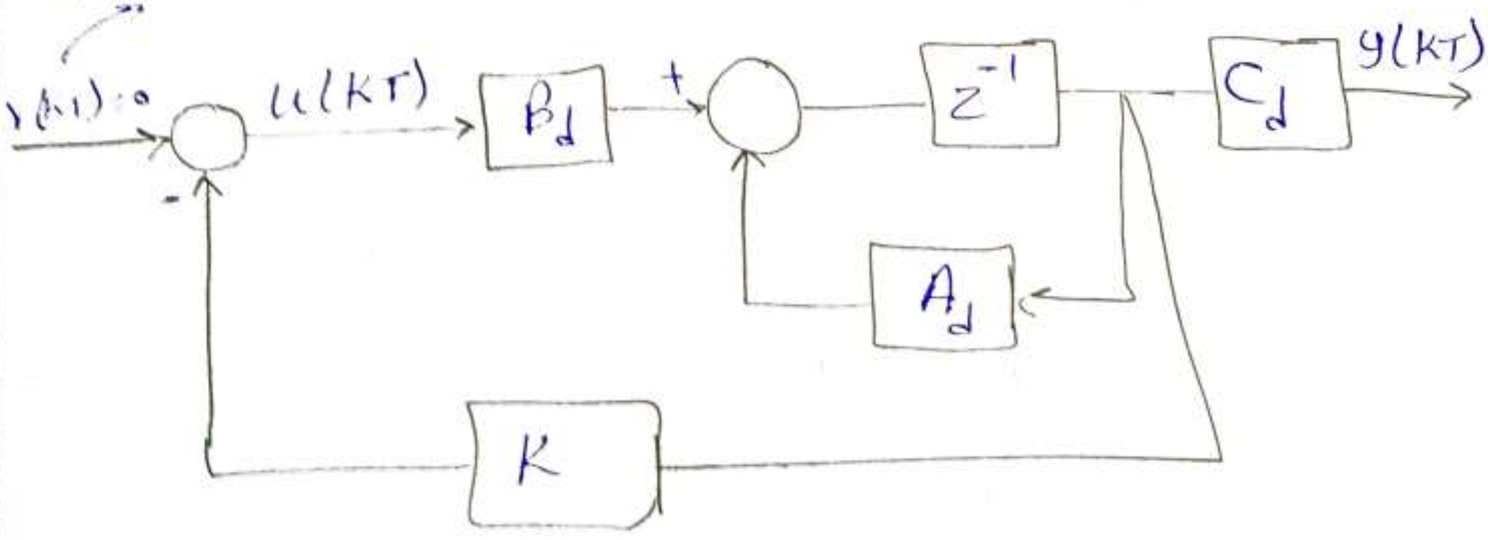
A, B, C → in discrete time System.



→ using state feedback control:-

$$\boxed{u(KT) = -K\,x(KT)}$$

regulation problem



→ The design Problem for the state feedback Control is to find the gain matrix K

where $K = [K_1 \quad K_2 \ldots K_n]$

where n is system order.

Given the desired Poles locations, we Can obtain the desired ch. equation :-

- assume the desired Poles located at $Z_1, Z_2, \ldots$

- the desired ch. equation $\alpha_c(z)$

$$\boxed{\alpha_c(z) = (z - z_1)(z - z_2) \ldots = 0} \rightarrow \boxed{1}$$

using the relation

$$u(KT) = -K\, x(KT)$$

$$x(K+1)T = A_d\, x(KT) + B_d\, \underbrace{u(KT)}_{\to -K\,x(KT)}$$

$$= (A_d - B_d K)\, x(KT)$$

$$\boxed{\text{ch. eq: } = \left| zI - A_d + B_d K \right| = 0} \longrightarrow \boxed{2}$$

eq. 1 $\simeq$ eq. 2

→ Compare coefficient to get K

Another method to get the gain matrix K:-

→ Ackermann's method

$$K = [K_1 \quad K_2 \dots K_n] = (0 \quad 0 \dots 1)\, M_c^{-1}\, \alpha_c(A)$$

wher:

$$M_c = (B \quad AB \quad A^2B \dots A^{n-1}B) \to \text{Controllability matrix}$$

$$\alpha_c(A) = \alpha_c(z)\Big|_{z \to A} \quad \text{where } \alpha_c(z)\text{: desired ch. equation}$$

For $n=2$

$$K = [K_1 \quad K_2] = (0 \quad 1)\, M_c^{-1}\, \alpha_c(A_d)$$

$$M_c = (B_d \quad A_d B_d)$$

→ if we give the desired poles in s-domain or in Cont. time system :-

given desired specs :-

$$\zeta \qquad \omega_n$$

$$s_{1,2} = -\zeta\omega_n \pm J\,\omega_n\sqrt{1-\zeta^2}$$

$$s = \sigma + J\omega \Longrightarrow z = r\angle\theta$$

$$z = r\cos\theta \pm Jr\sin\theta$$

$$z_{1,2} = r\angle\pm\theta$$

$r = e^{-\zeta \omega_n T}$ & $\theta = \omega_n T \sqrt{1-\zeta^2}$

$\boxed{EX}$ $x(K+1) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} x(K) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} u(K)$

$y(K) = (1 \quad 0)\, x(K)$

→ using state-feedback control to find the gain matrix K such that the desired poles located at :

$Z_{1,2} = 0.528 \pm J\, 0.295$

<u>using Ackermann</u>

$K = (0 \quad 1)\, M_c^{-1}\, \alpha_c(A)$

$\alpha_c(Z) = (Z - \underbrace{0.528}_{x} + \underbrace{J\,0.295}_{y})(Z - 0.528 - \underbrace{J\,0.295}_{y}) = 0$

$= (Z - (0.528)^2) + (0.295)^2 = 0$

$$\alpha_c(z) = z^2 - 1.056\, z + 0.366$$

$$\alpha_c(A) = \alpha_c(z)\Big|_{z=A} = A^2 - \cancel{}\, 1.056\, A + 0.366\, I$$

$$\alpha_c(A) = \begin{pmatrix} 0.31 & 1.889 \\ 0 & 0.31 \end{pmatrix}$$

$$M_c = \begin{pmatrix} B_d & A_d B_d \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 2 & 2 \end{pmatrix} \qquad M_c^{-1} = \frac{1}{-8} \begin{pmatrix} 2 & -6 \\ 2 & 2 \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} M_c^{-1} \alpha_c(A)$$

$$K = \begin{pmatrix} 0.0775 & 0.3945 \end{pmatrix}$$

Ex

$$X((K+1)T) = \begin{pmatrix} 1 & 0.0952 \\ 0 & 0.905 \end{pmatrix} X(KT) + \begin{pmatrix} 0.00484 \\ 0.0952 \end{pmatrix} u(KT)$$

$$y(KT) = \begin{pmatrix} 1 & 0 \end{pmatrix} X(KT)$$

→ Find gain matrix K using the Pole Placement design that makes system has a damping ratio $\zeta = 0.46$ & time constant $\tau = 1$ sec & $T = 0.1$ sec

Given $\zeta = 0.46$

$$\tau = \frac{1}{\zeta \omega_n} = 1 \text{ sec} \Rightarrow \boxed{\omega_n = 2.174 \text{ rad/sec}}$$

$$S_{1,2} = -\zeta \omega_n \pm J \omega_n \sqrt{1-\zeta^2}$$

$$r = e^{-\zeta \omega_n T} = 0.905$$

$$\theta = \omega_n T \sqrt{1-\zeta^2} = 0.193 \text{ rad}$$

$$Z_{1,2} = r\cos\theta \pm J\, r\sin\theta$$

$$Z_{1,2} = 0.888 \pm J\ 0.1736$$

$$K = \begin{pmatrix} 0 & 1 \end{pmatrix} M_c^{-1}\ \alpha_c(A)$$

$$M_c = \begin{pmatrix} B_d & A_d B_d \end{pmatrix} = \begin{pmatrix} 0.00484 & 0.0139 \\ 0.0952 & 0.0862 \end{pmatrix}$$

$$M_c^{-1} = \begin{pmatrix} -95.035 & 15.34 \\ 105.1 & -5.34 \end{pmatrix}$$

$$\alpha_c(z) = z^2 - 1.776z + 0.819 = 0$$

$$\alpha_c(A) = \begin{pmatrix} 0.043 & 0.0123 \\ 0 & 0.0307 \end{pmatrix}$$

$$K = \begin{pmatrix} 4.515 & 1.125 \end{pmatrix}$$